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On the first homology of the group of equivariant Lipschit homeomorphisms of the plane with circle action

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§1. Introduction and statement of the result

Let $L_G(M)$ denote the group of equivariant Lipschitz homeomorphisms of a G-manifold M which are isotopic to the identity through equivariant Lipschitz homeomorphisms with compact supports. In the previous papers [AF3],[AF4], we treated the subgroup $\mathcal{H}_{LIP,G}(M)$ of $L_G(M)$ whose elements are isotopic to the identity with respect to the compact open Lipschitz topology, and proved that $\mathcal{H}_{LIP,G}(M)$ is perfect when M is a principal G-manifold or M is a smooth G-manifold for a finite group G.

In this paper we consider the case of the complex plain $\mathbf{C}$ with canonical $U(1)$-action. We shall prove that the group $L_{U(1)}(\mathbf{C})$ is not perfect by calculating the the first homology group $H_1(L_{U(1)}(\mathbf{C}))$ which is defined as the quotient of $L_{U(1)}(\mathbf{C})$ by its commutator subgroup.

Let $\mathcal{C}(\mathbf{R})$ be the set of real valued functions f on $(0,1]$ such that there exists a positive number M satisfying

$$|f(x)-f(y)| \leq \frac{M}{x}(y-x) \quad \text{for } 0<x\leq y\leq 1.$$

Then $\mathcal{C}(\mathbf{R})$ is a vector space over $\mathbf{R}$. Let $\mathcal{C}_0(\mathbf{R})$ denote the subspace of those $f\in\mathcal{C}(\mathbf{R})$ with f bounded on $(0,1]$. Then we shall prove the following.

Theorem 1

$$H_1(L_{U(1)}(\mathbf{C})) \cong \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}).$$

Here the isomorphism is induced from the map assigning each $h \in L_{U(1)}(\mathbf{C})$ a function $\hat{a}_h \in \mathcal{C}(\mathbf{R})$ which stand for the degree of rotation of h as the point tend to zero (see §2). We note that the group $\mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R})$ is fairly large group since it contains linearly independent family of elements parameterized by $(0,1]$.

The situation is quite different in smooth category. Let $D_{U(1)}(\mathbf{C})$ denote the group of equivariant diffeomorphism group of $\mathbf{C}$ which are equivariantly diffeomorphic to the identity through compact supports. By [AF2], Theorem 3.2, we have that there exists an isomorphism $H_1(D_{U(1)}(\mathbf{C})) \cong \mathbf{R} \times \mathbf{U(1)}$ induced from the map assigning each $h \in D_{U(1)}(\mathbf{C})$ the differential of h at 0. Then it follows from Theorem 1 that the group $D_{U(1)}(\mathbf{C})$ is contained in the commutator subgroup of $L_{U(1)}(D)$, which implies that the first homology group of $D_{U(1)}(\mathbf{C})$ detect absolutely different geometric property.

§2. Orbit preserving equivariant Lipschitz homeomorphisms

Let D denote the unit disc in C and $L_{U(1)}(D)$ denote the group of $U(1)$-equivariant Lipschitz homeomorphisms of D which are isotopic to the identity through $U(1)$-equivariant homeomorphisms with identity on the boundary ∂D. Since $U(1)$ acts freely except for the origin, by combining Theorem 5.1 with Corollary 5.5 in [AF3], the group $H_1(L_{U(1)}(\mathbf{C}))$ is isomorphic to $H_1(L_{U(1)}(D))$.

Let $L([0,1])$ denote the group of Lipschitz homeomorphisms of the unit interval $[0,1]$ which are isotopic to the identity through Lipschitz homeomorphisms. Then we have a group homomorphism $P: L_{U(1)}(D) \to L([0,1])$ given by

$$P(h)(x) = |h(x)| \quad \text{for} \quad h \in L_{U(1)}(D),\ x \in [0,1].$$

There exists a right inverse $\Psi: L([0,1]) \to L_{U(1)}(D)$ of P defined by

$$\Psi(f)(xz) = f(x)z \quad \text{for} \quad f \in L([0,1]),\ x \in [0,1],\ z \in U(1).$$

Note that the kernel $KerP$ of P coincides with the set of those $h \in L_{U(1)}(D)$ which are orbit preserving. Next we shall investigate the relation between the groups $KerP$ and $\mathcal{C}(\mathbf{R})$.

For $h \in KerP$, let $a_h : (0,1] \to U(1)$ be the map satisfying

$$h(xz) = xza_h(x) \quad \text{for} \quad x \in (0,1],\ z \in U(1).$$

Now we investigate the properties of those maps a_h. For a map $\alpha : (0,1] \to U(1) \subset \mathbf{C}$, we define maps $\bar{\alpha} : [0,1] \to D$ and $F_\alpha : D \to D$ as follows.

$$\bar{\alpha}(x) = \begin{cases} x\alpha(x) & (0 < x \leq 1) \\ 0 & (x = 0) \end{cases},$$

$$F_\alpha(xz) = z\bar{\alpha}(x) \qquad (0 \leq x \leq 1,\ z \in U(1)).$$

Lemma 2 *The following conditions* (1), (2) *and* (3) *are equivalent.*
(1) *There exists a positive number* K *such that*

$$|\alpha(x) - \alpha(y)| \leq \frac{K}{x}(y - x) \quad \textit{for} \quad 0 < x \leq y \leq 1.$$

(2) $\bar{\alpha}$ *is a Lipschitz map.*
(3) F_α *is a Lipschitz map.*

Proof. First assume the condition (1). Then, for $0 < x \leq y \leq 1$, we have

$$|\bar{\alpha}(x) - \bar{\alpha}(y)| \leq x|\alpha(x) - \alpha(y)| + |\alpha(y)||x - y| \leq (K + 1)|x - y|.$$

Since $|\bar{\alpha}(x)| \leq x$ for $0 < x \leq 1$, the condition (2) is satisfied.

Secondly assume the condition (2). Then, for $0 < x \leq y \leq 1$, $z_1, z_2 \in U(1)$,

$$\begin{aligned} |F_\alpha(xz_1) - F_\alpha(yz_2)| &\leq |z_1(\bar{\alpha}(x) - \bar{\alpha}(y)| + |(z_1 - z_2)\bar{\alpha}(y)| \\ &\leq M(|x - y| + |z_1(y - x) + (z_1 x - z_2 y)|) \\ &\leq 3M|xz_1 - yz_2|, \end{aligned}$$

where M is a Lipschitz constant of $\bar{\alpha}$. Since $|F_\alpha(xz)| \leq M|xz|$, the condition (3) is satisfied.

Finally assume the condition (3). Then, for $0 < x \leq y \leq 1$, we have

$$\begin{aligned} |\alpha(x) - \alpha(y)| &\leq \frac{1}{x}(|x\alpha(x) - y\alpha(y)| + |(y - x)\alpha(y)|) \\ &= \frac{1}{x}(|F_\alpha(x) - F_\alpha(y)| + |y - x|) \leq \frac{L + 1}{x}(y - x), \end{aligned}$$

where L is a Lipschitz constant of F_α. Thus the condition (1) is satisfied and Lemma 2 follows.

Let $E : \mathbf{R} \to U(1)$ denote the exponential map given by $E(x) = e^{\sqrt{-1}x}$. Let $h \in KerP$. Since h is identity on ∂D, $a_h(1) = 1$. Let $\hat{a}_h : (0,1] \to \mathbf{R}$ be the lifting of a_h for E with $\hat{a}_h(1) = 0$. Then $E \circ \hat{a}_h = a_h$.

Lemma 3 *$\hat{a}_h$ is contained in $\mathcal{C}(\mathbf{R})$. Conversely if $\hat{\alpha} \in \mathcal{C}(\mathbf{R})$, then $E \circ \hat{\alpha}$ satisfies the condition* (1) *in* Lemma 2.

Proof By Lemma 2, there exists a positive number K such that

$$|a_h(x) - a_h(y)| \leq \frac{K}{x}(y - x) \quad for \quad 0 < x \leq y \leq 1.$$

Note that, for each $x, y \in (0, 1]$ with $x < y$, the restriction $a_h \big|_{[x,y]}$ is Lipschitz. Then we can choose an increasing series of points $x = x_0 < x_1 < \cdots < x_{n-1} < x_n = y$ such that

$$|a_h(x_{i-1}) - a_h(x_i)| \leq \sqrt{3} \quad (i = 1, ..., n).$$

It follows that

$$|\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)| \leq \frac{2\pi}{3} \quad (i = 1, ..., n).$$

Then we have

$$\begin{aligned} |a_h(x_{i-1}) - a_h(x_i)| &= |e^{\sqrt{-1}\,\hat{a}(x_{i-1})} - e^{\sqrt{-1}\,\hat{a}(x_i)}| \\ &= 2\left|\sin \frac{\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)}{2}\right| \\ &= \left|\cos \frac{\theta\left(\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)\right)}{2}\right| |\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)|, \end{aligned}$$

for some $0 < \theta < 1$. Thus

$$|\hat{a}_h(x_{i-1}) - \hat{a}_h(x_i)| \leq 2\ |a_h(x_{i-1}) - a_h(x_i)| \leq \frac{2K}{x_{i-1}}|x_{i-1} - x_i|.$$

Therefore we have

$$|\hat{a}_h(x) - \hat{a}_h(y)| \leq \sum_{i=1}^{n} \frac{2K}{x_{i-1}}|x_{i-1} - x_i| \leq \frac{2K}{x}(y - x),$$

and then we have that $\hat{a}_h \in \mathcal{C}(\mathbf{R})$.

Since

$$|E(x) - E(y)| = |e^{\sqrt{-1}x} - e^{\sqrt{-1}y}| \leq (y - x) \quad \text{for} \quad 0 < x \leq y \leq 1,$$

it is clear that, for each $\hat{\alpha} \in \mathcal{C}(\mathbf{R})$, $E \circ \hat{\alpha}$ satisfies the condition (1) in Lemma 2. This completes the proof of Lemma 3.

§3. Basic homomorphisms

By Lemma 3 we can define a homomorphism

$$T: \ Ker P \to \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}), \qquad T(h) = \hat{a}_h \mod \mathcal{C}_0(\mathbf{R}).$$

Now we have a map

$$\Theta: \ L_{U(1)}(D) \to L([0,1]) \times \mathcal{C}/\mathcal{C}_0$$

defined by

$$\Theta(h) = (P(h),\ T(\Psi(P(h))^{-1} \circ h)).$$

Proposition 4 *Θ is an onto group homomorphism.*

Proof. First we prove that Θ is a group homomorphism. For each $h \in L_{U(1)}(D)$, we set $\tilde{h} = \Psi(P(h))^{-1} \circ h$. Let $h_i \in L_{U(1)}(D)$ $(i = 1,2)$. Since P is a group homomorphism, in order to prove Θ a group homomorphism it is sufficient to prove that

$$\hat{a}_{\widetilde{h_1 \circ h_2}} = \hat{a}_{\tilde{h}_1} + \hat{a}_{\tilde{h}_2} \quad \mod \ \mathcal{C}_0(\mathbf{R}).$$

If $0 < x \ \leq 1,\ z \in U(1)$, then

$$h_i(xz) = P(h_i)(x)\, z a_{\tilde{h}_i}(x)^{-1} \quad (i = 1,2),$$

and

$$(h_1 \circ h_2)(xz) = P(h_1 \circ h_2)(x)\, z a_{\widetilde{h_1 \circ h_2}}(x)^{-1}.$$

On the other hand we have

$$(h_1 \circ h_2)(xz) = P(h_1 \circ h_2)(x)\, z a_{\tilde{h}_2}(x)^{-1} a_{\tilde{h}_1}(P(h_2)(x))^{-1}.$$

Then

$$a_{\widetilde{h_1 \circ h_2}} = (a_{\tilde{h}_1} \circ P(h_2)) \cdot a_{\tilde{h}_2}.$$

Thus

$$\hat{a}_{\widetilde{h_1 \circ h_2}} = \hat{a}_{\tilde{h}_1} \circ P(h_2) + \hat{a}_{\tilde{h}_2}.$$

Let M and M' be Lipschitz constants of $P(h_2)$ and $P(h_2)^{-1}$, respectively. Let $x \in (0,1]$. For the case $x \leq P(h_2)(x)$, by Lemma 3 there exists a positive number K such that

$$|\hat{a}_{\tilde{h}_1}(P(h_2)(x)) - \hat{a}_{\tilde{h}_1}(x)| \leq \frac{K}{x}|P(h_2)(x) - x| \leq K(M+1).$$

By definition $x \leq M' P(h_2)(x)$. Then, for the case $P(h_2)(x) < x$, we have

$$|\hat{a}_{\tilde{h}_1}(P(h_2)(x)) - \hat{a}_{\tilde{h}_1}(x)| \leq \frac{K}{P(h_2)(x)}|P(h_2)(x) - x| \leq K(1+M').$$

Then we have

$$\hat{a}_{\tilde{h}_1} \circ P(h_2) - \hat{a}_{\tilde{h}_1} \in \mathcal{C}_0(\mathbf{R}).$$

Thus

$$\hat{a}_{\widetilde{h_1 \circ h_2}} = \hat{a}_{\tilde{h}_1} + \hat{a}_{\tilde{h}_2} \quad \mod \ \mathcal{C}_0(\mathbf{R}).$$

Therefore Θ is a group homomorphism.

Let $f \in L([0,1])$, $\hat{\alpha} \in \mathcal{C}(\mathbf{R})$. Combining Lemma 2 with Lemma 3, we have that $F_{E\circ\hat{\alpha}} \in KerP$. Set

$$h(xz) = f(x)F_{E\circ\hat{\alpha}}(xz) \quad \text{for} \quad 0 \leq x \leq 1, \ z \in U(1).$$

Then we see that $h \in L_{U(1)}(D)$ and $\Theta(h) = (f, \ \hat{\alpha} \mod \mathcal{C}_0(\mathbf{R}))$. Thus Θ is onto. This completes the proof of Proposition 4.

§4 Proof of main theorem

Proposition 5 *$Ker\ \Theta$ is contained in the commutator subgroup of $L_{U(1)}(D)$.*

Proof. If $h \in Ker\ \Theta$, then $h \in KerP$ and $\hat{a}_h \in \mathcal{C}_0(\mathbf{R})$. Thus, for any positive number ε, there exists an integer n such that $\left|\frac{\hat{a}_h(x)}{n}\right| \leq \varepsilon$ for $0 < x \leq 1$ and

$$\left|\frac{\hat{a}_h(x)}{n} - \frac{\hat{a}_h(y)}{n}\right| \leq \frac{\varepsilon}{x}(y-x) \quad \text{for} \quad 0 < x \leq y \leq 1.$$

Note that $a_h = E(n\hat{a}_h) = E(\hat{a}_h)^n$. Then, for a sufficiently small positive number ε, we can assume that $|\hat{a}_h(x)| \leq \varepsilon$ for $0 < x \leq 1$ and

$$|\hat{a}_h(x) - \hat{a}_h(y)| \leq \frac{\varepsilon}{x}(y-x) \quad \text{for} \quad 0 < x \leq y \leq 1.$$

Let v be a real valued smooth monotone increasing function on $(0,1]$ such that

$$v(x) = \begin{cases} \log x & (0 < x \le 1/2), \\ 0 & (3/4 \le x \le 1). \end{cases}$$

Then it is easy to see $v \in \mathcal{C}(\mathbf{R})$. Let f be a real valued function on $[0,1]$ defined by

$$f(x) = \begin{cases} xe^{\hat{a}_h(x)} & (0 < x \le 1), \\ 0 & (x = 0). \end{cases}$$

Note that $f(1) = 1$. We shall prove that $f \in L([0,1])$ for sufficiently small ε. If $0 < x \le y \le 1$, then we have

$$\begin{aligned} & |(f(y) - y) - (f(x) - x)| \\ = \; & |(y-x)(e^{\hat{a}_h(y)} - 1) + x(e^{\hat{a}_h(y)} - e^{\hat{a}_h(x)})| \\ \le \; & (y-x)|e^{|\hat{a}_h(y)|} - 1| + x|\hat{a}_h(y) - \hat{a}_h(x)|e^{\hat{a}_h(x)+\theta(\hat{a}_h(y)-\hat{a}_h(x))} \\ \le \; & ((e^\varepsilon - 1) + \varepsilon e^{3\varepsilon})(y-x), \end{aligned}$$

for some $0 < \theta < 1$. Here we take the positive number ε satisfying

$$(e^\varepsilon - 1) + \varepsilon e^{3\varepsilon} < 1.$$

Then it follows from [AF3], Lemma 4.1 that the function f is a Lipschitz homeomorphism of $[0,1]$ which is isotopic to the identity through Lipschitz homeomorphisms.

If $0 < x \le \dfrac{1}{2e^\varepsilon}$, then we have

$$v(f(x)) - v(x) = \log(xe^{\hat{a}_h(x)}) - \log x = \hat{a}_h(x).$$

Then, for $0 < x \le \dfrac{1}{2e^\varepsilon}$, $z \in U(1)$ we have

$$\begin{aligned} (F_{Eov}^{-1} \circ \Psi(f)^{-1} \circ F_{Eov}^{-1} \circ \Psi(f))(xz) &= (F_{Eov}^{-1} \circ \Psi(f)^{-1} \circ F_{Eov}^{-1})(f(x)z) \\ &= (F_{Eov}^{-1} \circ \Psi(f)^{-1})(f(x)ze^{\sqrt{-1}\,v(f(x))}) \\ &= F_{Eov}^{-1}(xze^{\sqrt{-1}\,v(f(x))}) \\ &= xze^{\sqrt{-1}\,v(f(x))}e^{-\sqrt{-1}\,v(x)} \\ &= h(xz) \end{aligned}$$

Set

$$g = h \circ \Psi(f)^{-1} \circ F_{Eov}^{-1} \circ \Psi(f) \circ F_{Eov}.$$

$$g(xz) = xz \quad \text{for} \quad 0 \leq x \leq \frac{1}{2e^{\varepsilon}}, \ z \in U(1).$$

Thus the support of g is contained in $D \backslash \{0\}$. From [AF3], Theorem 5.1, g is contained in the commutator subgroup of $L_{U(1)}(D)$. Hence h is also contained in the commutator subgroup. This completes the proof of Proposition 5.

Proof of Theorem 1. Let $\iota : Ker\Theta \to L_{U(1)}(D)$ denote the inclusion. By Proposition 4 we have the following exact sequence.

$$\begin{aligned} Ker\Theta/[Ker\Theta, L_{U(1)}(D)] &\xrightarrow{\iota_*} H_1(L_{U(1)}(D)) \\ &\xrightarrow{\Theta_*} H_1(L([0,1]) \times \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R})) \to 1. \end{aligned}$$

Since $\iota_* = 0$ by Proposition 5, Θ_* is isomorphic. By [TS], [AF4], the group $L([0,1])$ is perfect. Thus we have

$$H_1(L_{U(1)}(D)) \cong \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}).$$

Remark. Let v_c $(0 < c \leq 1)$ be real valued smooth functions on $(0,1]$ such that

$$v_c(x) = \begin{cases} (-\log x)^c & (0 < x \leq 1/2), \\ 0 & (3/4 \leq x \leq 1). \end{cases}$$

Then $v_c \in \mathcal{C}(\mathbf{R})$. Thus the group $\mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R})$ contains linearly independent families $\{v_c \mod \mathcal{C}_0 \ ; \ 0 < c \leq 1\}$.

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